The Laplace Transform.
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Introduction:

* Use it to solve ODE/PDE

* Applications:
  - Analysis of electrical circuits.
  - NMR (Nuclear magnetic resonance), Spectroscopy etc
  - Signal processing.

* Basics:

Converts time (t) domain to a frequency (ω, s) domain status.

* Notations:

$\mathcal{L}\{f(t)\} = F(s)$ — Laplace transform

$\mathcal{L}^{-1}\{F(s)\} = f(t)$ — Inverse transform

Start with

$f(t) \xrightarrow{\mathcal{L}} F(s)$: Laplace transform

$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t)$: Inverse transform

* Definition

The Laplace transform of a $f(t)$ is

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Conditions
$t \geq 0$
$s > 0$

exponential order function.

Improper integral

Note: For the integral to exist, we need $t \geq 0$ and also the function f have to be of exponential order.

Example 1.

Find the Laplace transform of $f(t) = 1$

Solution

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\,dt$$

NB
$$\int e^{kt}\,dt = \frac{1}{k} e^{kt} + C$$

$$= \lim_{n \to \infty} \int_0^n e^{-st}\,dt$$

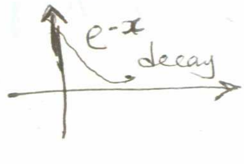
$$= \lim_{n \to \infty} \frac{e^{-st}}{-s}\Big|_0^n$$

$$= \lim_{n \to \infty} \left[\frac{e^{-sb}}{-s} + \frac{1}{s}\right]$$

tends to zero (0)

$$= 1/s \quad (\text{if } s > 0)$$

NB



Therefore

$$\mathcal{L}\{1\} = 1/s$$

Example 2.

Find the Laplace transform of $f(t) = e^{at}$

Solution

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt$$

$$= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty$$

($a - s > 0 \Rightarrow$ no limit
i.e $a > s$
$a - s < 0$ i.e $a < s$)

$$= \frac{1}{a-s}[0 - 1]$$

$$= \frac{-1}{a-s} = \frac{1}{s-a}$$

$$\Rightarrow \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad s > a$$

Note

$\mathcal{L}\{1\} = \mathcal{L}\{e^{0t}\}$ i.e $a=0$

$= \frac{1}{s} \quad s>0$

Example 3

Find the Laplace transform of

$f(t)=t$

Solution

$L\{t\} = \int_0^\infty e^{-st} \cdot t\,dt$ — you'll need to use integration by part.

Recall/digression: Integration by part.

① $\int x e^x dx$ let

$\int u\,dv = uv - \int v\,du$

$u = x$
$du = 1dx$
$v = e^x$
$dv = e^x dx$

$= xe^x - \int e^x (1dx)$

$= xe^x - e^x + c$

② $\int x \sin x\,dx$

$\int u\,dv = uv - \int v\,du$

$u = x$
$du = 1dx$
$v = -\cos x$
$dv = \sin x\,dx$

$= x(-\cos x) - \int -\cos x (1) dx$

$= -x\cos x + \int \cos x\,dx$

$= -x\cos x + \sin x + c$

③ $\int x^2 \ln x\,dx$

$\int u\,dv = uv - \int v\,du$

$u = \ln x$
$du = \frac{1}{x}$
$v = \frac{x^3}{3}$
$dv = x^2 dx$

$= \ln x\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx$

$= \frac{x^3}{3}\ln x - \frac{1}{3} \cdot \frac{x^3}{3} + c = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c$

End of digression

②

$= \frac{-e^{-st} \cdot t}{s}\Big|_0^\infty - \int_0^\infty \frac{-e^{-st}}{s} \times 1\,dt$

$u = t$
$du = 1dt$
$v = \frac{e^{-st}}{-s}$
$dv = e^{-st}dt$

$= -\frac{t e^{-st}}{s}\Big|_0^\infty + \int \frac{e^{-st}}{s} dt$

$= 0 + \frac{e^{-st}}{-s^2}\Big|_0^\infty = 0 - \frac{1}{-s^2}$

$= \frac{1}{s^2}$

NB:

$L\{t^n\} = \frac{n!}{s^{n+1}}$ [Prove this]

Example 4

Find the Laplace transform of

$f(t) = \sin at$

Soln

$L\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt$

$\int u\,dv = uv - \int v\,du$

$u = e^{-st}$
$du = -\frac{e^{-st}}{s} dt$
$v = -a\cos at$
$dv = \sin at$

...

NB $\int e^{at} \sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

set $a=-s$ and $b=a$

$\Rightarrow L\{\sin at\} = \int_0^\infty e^{-st} \sin at\,dt$

$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$

$= \left[\frac{e^{-\infty}}{s^2+a^2}(-s\sin\infty - a\cos\infty)\right] - \left[\frac{e^{-0}}{s^2+a^2}(-s\sin 0 - a\cos 0)\right]$

(NB $e^{-\infty}=0$, $e^0 = e^{-0} = 1$)

$= 0 - \frac{1}{s^2+a^2}(0 - a) = \frac{a}{s^2+a^2}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

quiz 1

Show that

$$\int e^{-st}\sin at\,dt = \frac{e^{-st}}{s^2+a^2}\left(-s\sin at - a\cos at\right)+C$$

Example 5

$L\{\cos at\}$?

Soln.

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ – By definition.

$\therefore L\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$ (– Use Integration by part)

NB: $\int e^{at}\cos bt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$

where $a=-s$ and $b=a$ ...

$\therefore L\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

$e^{-\infty}=0,\ e^0=1,\ \sin 0 = 0\ \ \cos 0 = 1$

$$= \left[\frac{e^{-\infty}}{s^2+a^2}(-s\cos\infty + a\sin\infty)\right] - \left[\frac{e^0}{s^2+a^2}(-s\cos 0 + a\sin 0)\right]$$

$$= 0 - \frac{1}{s^2+a^2}(-s)$$

$$= \frac{s}{s^2+a^2}$$

$$\therefore L\{\cos at\} = \frac{s}{s^2+a^2}$$

Quize 2

Find the laplace transform of

a) $f(t) = \cosh at$ $\left[\text{Hint } \cosh\theta = \frac{e^\theta + e^{-\theta}}{2},\ \ L\{\cosh at\} = \frac{s}{s^2-a^2}\right]$

b) $f(t) = \sinh at$

Hint $\sinh\theta = \frac{e^\theta - e^{\theta}}{2}$

$L\{\sinh at\} = \frac{a}{s^2-a^2}$

Laplace as Linear Operator/Property/functio

$$L\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-st}(c_1 f(t) + c_2 g(t))dt$$

$$= \int_0^\infty [c_1 e^{-st} f(t) + c_2 e^{-st} g(t)]dt$$

$$= c_1\int_0^\infty e^{-st} f(t)\,dt + c_2\int_0^\infty e^{-st} g(t)\,dt$$

$$= c_1 L\{f(t)\} + c_2 L\{g(t)\}$$

Laplace Transform of Derivatives

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Use Integration by part.

$\int uv' = uv - \int u'v$

$u = e^{-st}$

$u' = -se^{-st}$

$v' = f'(t)$

$v = f(t)$

$$= \left[e^{-st}f(t)\right]_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt$$

$$= \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt$$

$\left(e^{-\infty}=0,\ e^{-0}=1 \text{ and } L\{f(t)\} = \int_0^\infty e^{-st}f(t)\,dt\right)$

$$= 0 - f(0) + sL\{f(t)\}$$

Therefore:

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Also

$$L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$

$$= s(sL\{f(t)\} - f(0)) - f'(0)$$

$$= s^2L\{f(t)\} - sf(0) - f'(0)$$

Quiz 3 What is $L\{f'''(t)\}$?

L

Example 5

If $L\{f'(t)\} = sL\{f(t)\} - f(0)$ and $L\{\sin at\} = \frac{a}{s^2+a^2}$

What is $L\{\cos at\}$?

solution

if $f'(t) = \cos at$ then $f(t) = \frac{1}{a}\sin at$

then

$$L\{\cos at\} = sL\{\tfrac{1}{a}\sin at\} - \cancel{\tfrac{1}{a}\sin 0}^{\,0}$$

$$= \frac{s}{a}L\{\sin at\}$$

$$= \frac{s}{\cancel{a}} \cdot \frac{\cancel{a}}{s^2+a^2}$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

[easier than the integration by part approach!]

Laplace transform of polynomials

$L\{1\} = \frac{1}{s}$, $L\{f'(t)\} = sL\{f(t)\} - f(0)$

$\Rightarrow L\{f'\} = sL\{f\} - f(0)$

and $L\{f'(t)\} + f(0) = sL\{f(t)\}$

What is $L\{t\}$?

Soln

$$\cancel{L\{t\}} = L\{t\} = \frac{1}{s}(L\{1\}) - 0$$

$$= \frac{1}{s} \cdot \frac{1}{s}$$

$$= \frac{1}{s^2}$$

[i.e $L\{t\} = \frac{1}{s}L\{f'(t)\} + f(0)$]

Also

$$L\{t^2\} = \frac{1}{s}L\{2t\} = \frac{2}{s}L\{t\}$$

$$= \frac{2}{s} \cdot \frac{1}{s} = \frac{2}{s^3}$$

Also

$$L\{t^3\} = \frac{1}{s}L\{3t^2\} = \frac{3}{s}L\{t^2\}$$

$$= \frac{3 \cdot 2}{s^4}$$

Therefore

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

Solving (linear) differential equations using Laplace transform

$y'' + 5y' + 6y = 0$; $y(0) = 2$ $y'(0) = 3$

$$L\{y''\} + 5L\{y'\} + 6L\{y\} = L\{0\}$$

(Recall $L\{y'\} = sL\{y\} - y(0)$)

$$L\{y''\} = sL\{y'\} - y'(0)$$

$$= s(sL\{y\} - y(0)) - y'(0)$$

$$= s^2L\{y\} - sy(0) - y'(0)$$

$$= s^2L\{y\} - 2s - 3 \qquad \text{(i)}$$

$$5L\{y'\} = 5(sL\{y\} - y(0))$$

$$= 5sL\{y\} - 5y(0)$$

$$= 5sL\{y\} - 10 \qquad \text{(ii)}$$

$$6L\{y\} \qquad \text{(iii)}$$

add (i), (ii) & (iii).

$$\Rightarrow s^2L\{y\} - 2s - 3 + 5sL\{y\} - 10 + 6L\{y\} = 0$$

collect like terms

$$L\{y\}(s^2+5s+6) - 2s - 13 = 0$$

$$L\{y\}(s^2+5s+6) = 2s+13$$

$$L\{y\} = \frac{2s+13}{s^2+5s+6}$$

~~(take inverse laplace transform of both side to get y but lets not do that yet)~~

$$\cancel{L\{y\} = \frac{2s+13}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3}}$$

$$L\{y\} = \frac{2s+13}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)}$$

$$\frac{As+3A+Bs+2B}{(s+2)(s+3)} = \frac{2s+13}{(s+2)(s+3)}$$

$(A+B)s + 3A + 2B = 2s + 13$

$\Rightarrow A+B = 2$ & $3A+2B = 13$

$\Rightarrow A = 9$ & $B = -7$

$$\therefore L\{y\} = \frac{9}{s+2} - \frac{7}{s+3}$$

$$= 9\left(\frac{1}{s+2}\right) - 7\left(\frac{1}{s+3}\right)$$

Recall

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{y\} = 9L\{e^{-2t}\} - 7L\{e^{-3t}\}$$

$$= L\{9e^{-2t} - 7e^{-3t}\}$$

Take the inverse laplace transform of both sides

$$y = 9e^{-2t} - 7e^{-3t}$$

Multiplying functions by exponential to transform (laplace)

$$L\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)\,dt$$

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)\,dt$$

$\Rightarrow$

$$L\{e^{at}f(t)\} = F(s-a)$$

$$L\{\cos 2t\} = F(s) = \frac{s}{s^2+4}$$

$$L\{e^{3t}\cos 2t\} = F(s-3) = \frac{s-3}{(s-3)^2+4}$$

Using Laplace transform to solve a nonhomogeneous equation.

Example

$y'' + y = \sin 2t \qquad y(0) = 2, \; y'(0) = 1$

$\Rightarrow L\{y''\} + L\{y\} = L\{\sin 2t\}$

$L\{y''\} = s^2 L\{y\} - sy(0) - y'(0)$

$= s^2 L\{y\} - 2s - 1$ —— (i)

$L\{\sin 2t\} = \dfrac{2}{s^2+4}$ —— (ii) $\quad L\{\sin at\} = \dfrac{a}{s^2+a^2}$

Put (i) & (ii) in eqn.

$$s^2 L\{y\} - 2s - 1 + L\{y\} = \frac{2}{s^2+4}$$

$$L\{y\}(s^2+1) = \frac{2}{s^2+4} + 2s + 1$$

$$L\{y\} = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s}{(s^2+1)} + \frac{1}{(s^2+1)}$$

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$= \frac{(As+B)(s^2+1) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+1)}$$

$$= \frac{As^3 + As + Bs^2 + B + Cs^3 + 4Cs + Ds^2 + 4D}{(s^2+4)(s^2+1)}$$

$$= \frac{(A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D)}{(s^2+4)(s^2+1)}$$

$A+C=0 \quad B+D=0 \quad \Rightarrow C=0, \; A=0$

$A+4C=0 \quad B+4D=2 \quad B=-2/3, \; D=2/3$

$$L\{y\} = -\frac{1}{3}\left(\frac{2}{s^2+4}\right) + \frac{2}{3}\left(\frac{1}{s^2+1}\right) + \frac{2}{1}\left(\frac{s}{s^2+1}\right) + \frac{1}{s^2+1}$$

$$\left(L\{\sin at\} = \frac{a}{s^2+a^2} \quad , \quad L\{\cos at\} = \frac{s}{s^2+a^2} \right)$$

$$\Rightarrow L\{y\} = -\frac{1}{3} L\{\sin 2t\} + \frac{2}{3} L\{\sin t\} + 2\, L\{\cos t\}$$

$$+ L\{\sin t\}.$$

Take the inverse ~~$L^{-1}\{y\}$~~ laplace transform of all

$$y(t) = -\frac{1}{3}\sin 2t + \frac{2}{3}\sin t + 2\cos t + \sin t$$

$$= -\frac{1}{3}\sin 2t + \frac{5}{3}\sin t + 2\cos t$$